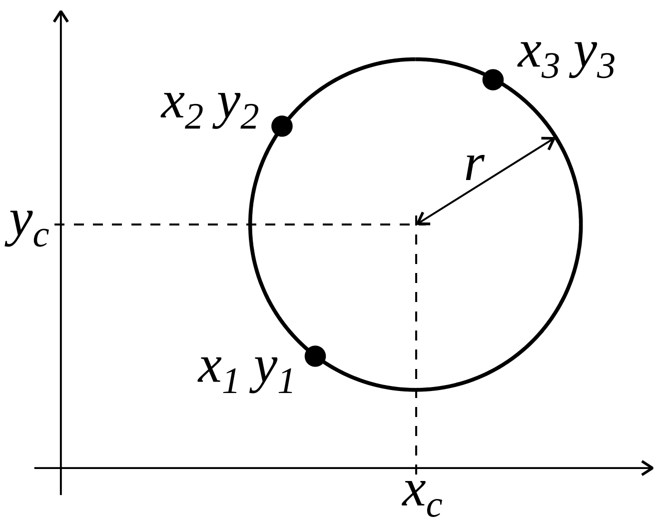
Calculation of Circle Centre-Point and Radius from Three Points on Circle Circumference



Known variables:

Coordinates of 3 points on circle circumference:

$x_1 y_1$, $x_2 y_2$ and $x_3 y_3$

Unknown variables:

$x_c y_c$ centre-point of circle

r radius of circle

(i) Initial Equations:

$$(x_1 - x_c)^2 + (y_1 - y_c)^2 = r^2$$

$$(x_2 - x_c)^2 + (y_2 - y_c)^2 = r^2$$

$$(x_3 - x_c)^2 + (y_3 - y_c)^2 = r^2$$

(ii) Expand initial equations:

$$x_1^2 - 2x_1x_c + x_c^2 + y_1^2 - 2y_1y_c + y_c^2 = r^2$$

$$x_2^2 - 2x_2x_c + x_c^2 + y_2^2 - 2y_2y_c + y_c^2 = r^2$$

$$x_3^2 - 2x_3x_c + x_c^2 + y_3^2 - 2y_3y_c + y_c^2 = r^2$$

(iii) Equate the r^2 of the 1st and 2nd expanded equations and re-arrange so y_c is alone:

$$x_1^2 - 2x_1x_c + x_c^2 + y_1^2 - 2y_1y_c + y_c^2 = x_2^2 - 2x_2x_c + x_c^2 + y_2^2 - 2y_2y_c + y_c^2$$

Cancel out x_c^2 and y_c^2 from each side:

$$x_1^2 - 2x_1x_c + y_1^2 - 2y_1y_c = x_2^2 - 2x_2x_c + y_2^2 - 2y_2y_c$$

Re-arrange so y_c is on one side:

$$x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2) = 2y_c \times (y_1 - y_2)$$

$$\therefore y_c = \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2)}{2 \times (y_1 - y_2)}$$

(iv) Similarly, equate the r^2 of the 1st and 3rd expanded equations and re-arrange so y_c is alone:

$$x_1^2 - 2x_1x_c + x_c^2 + y_1^2 - 2y_1y_c + y_c^2 = x_3^2 - 2x_3x_c + x_c^2 + y_3^2 - 2y_3y_c + y_c^2$$

$$\therefore x_1^2 - 2x_1x_c + y_1^2 - 2y_1y_c = x_3^2 - 2x_3x_c + y_3^2 - 2y_3y_c$$

$$\therefore x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2) = 2y_c \times (y_1 - y_2)$$

$$\therefore y_c = \frac{x_1^2 - x_3^2 + y_1^2 - y_3^2 - 2x_c \times (x_1 - x_3)}{2 \times (y_1 - y_3)}$$

(v) Equate the y_c from equations (iii) and (iv) and re-arrange so x_c is alone:

$$\frac{x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2)}{2 \times (y_1 - y_2)} = \frac{x_1^2 - x_3^2 + y_1^2 - y_3^2 - 2x_c \times (x_1 - x_3)}{2 \times (y_1 - y_3)}$$

Cancel out the 2's in the denominators on each side:

$$\frac{x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2)}{(y_1 - y_2)} = \frac{x_1^2 - x_3^2 + y_1^2 - y_3^2 - 2x_c \times (x_1 - x_3)}{(y_1 - y_3)}$$

Cross-multiply each side:

$$(x_1^2 - x_2^2 + y_1^2 - y_2^2) \times (y_1 - y_3) - 2x_c \times (x_1 - x_2) \times (y_1 - y_3)$$
$$= (x_1^2 - x_3^2 + y_1^2 - y_3^2) \times (y_1 - y_2) - 2x_c \times (x_1 - x_3) \times (y_1 - y_2)$$

Re-arrange so x_c is on one side:

$$(x_1^2 - x_2^2 + y_1^2 - y_2^2) \times (y_1 - y_3) - (x_1^2 - x_3^2 + y_1^2 - y_3^2) \times (y_1 - y_2)$$
$$= 2x_c \times [(x_1 - x_2) \times (y_1 - y_3) - (x_1 - x_3) \times (y_1 - y_2)]$$

$$\therefore x_c = \frac{(x_1^2 - x_2^2 + y_1^2 - y_2^2) \times (y_1 - y_3) - (x_1^2 - x_3^2 + y_1^2 - y_3^2) \times (y_1 - y_2)}{2 \times [(x_1 - x_2) \times (y_1 - y_3) - (x_1 - x_3) \times (y_1 - y_2)]}$$

(vi) Repeat steps (iii), (iv) and (v) but getting x_c alone first and y_c alone last to get:

$$y_c = \frac{(y_1^2 - y_2^2 + x_1^2 - x_2^2) \times (x_1 - x_3) - (y_1^2 - y_3^2 + x_1^2 - x_3^2) \times (x_1 - x_2)}{2 \times [(y_1 - y_2) \times (x_1 - x_3) - (y_1 - y_3) \times (x_1 - x_2)]}$$

(vii) Calculate r (any of these three equations can be used):

$$r = \sqrt{(x_1 - x_c)^2 + (y_1 - y_c)^2} = \sqrt{(x_2 - x_c)^2 + (y_2 - y_c)^2} = \sqrt{(x_3 - x_c)^2 + (y_3 - y_c)^2}$$

Notes:

The equations in steps (v) and (vi) for x_c and y_c share common factors:

$N_2 = (x_1^2 - x_2^2 + y_1^2 - y_2^2)$

$D_1 = (x_1 - x_2) \times (y_1 - y_3) - (x_1 - x_3) \times (y_1 - y_2)$

$N_3 = (x_1^2 - x_3^2 + y_1^2 - y_3^2)$

Therefore the final equations in steps (v) and (vi) can be simplified:

$$x_c = \frac{N_2 \times (y_1 - y_3) - N_3 \times (y_1 - y_2)}{2 \times D_1}$$

$$y_c = \frac{N_2 \times (x_1 - x_3) - N_3 \times (x_1 - x_2)}{-2 \times D_1} = \frac{N_3 \times (x_1 - x_2) - N_2 \times (x_1 - x_3)}{2 \times D_1}$$

If the three points are on the same line then $D_1 = 0$ because there is no circle (it has infinite radius).

If two or more of the three points are the same, then $N_2 = 0$ and/or $N_3 = 0$ and there won't be a solution because three different points are required to calculate a circle's centre-point and radius.