Calculation of Circle Centre-Point and Radius from Three Points on Circle Circumference

Known variables:

Coordinates of 3 points on circle circumference:
\(x_1y_1, x_2y_2, \text{ and } x_3y_3\)

Unknown variables:

\(x_cy_c\) centre-point of circle

\(r\) radius of circle

(i) Initial Equations:

\[(x_1 - x_c)^2 + (y_1 - y_c)^2 = r^2\]

\[(x_2 - x_c)^2 + (y_2 - y_c)^2 = r^2\]

\[(x_3 - x_c)^2 + (y_3 - y_c)^2 = r^2\]

(ii) Expand initial equations:

\[x_1^2 - 2x_1x_c + x_c^2 + y_1^2 - 2y_1y_c + y_c^2 = r^2\]

\[x_2^2 - 2x_2x_c + x_c^2 + y_2^2 - 2y_2y_c + y_c^2 = r^2\]

\[x_3^2 - 2x_3x_c + x_c^2 + y_3^2 - 2y_3y_c + y_c^2 = r^2\]

(iii) Equate the \(r^2\) of the 1st and 2nd expanded equations and re-arrange so \(y_c\) is alone:

\[x_1^2 - 2x_1x_c + x_c^2 + y_1^2 - 2y_1y_c + y_c^2 = x_2^2 - 2x_2x_c + x_c^2 + y_2^2 - 2y_2y_c + y_c^2\]

Cancel out \(x_c^2\) and \(y_c^2\) from each side:

\[x_1^2 - 2x_1x_c + y_1^2 = x_2^2 - 2x_2x_c + y_2^2\]

Re-arrange so \(y_c\) is on one side:

\[x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2) = 2y_c \times (y_1 - y_2)\]

\[y_c = \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2x_c \times (x_1 - x_2)}{2 \times (y_1 - y_2)}\]

(iv) Similarly, equate the \(r^2\) of the 1st and 3rd expanded equations and re-arrange so \(y_c\) is alone:

\[x_1^2 - 2x_1x_c + x_c^2 + y_1^2 - 2y_1y_c + y_c^2 = x_3^2 - 2x_3x_c + x_c^2 + y_3^2 - 2y_3y_c + y_c^2\]

\[x_1^2 - 2x_1x_c + y_1^2 = x_3^2 - 2x_3x_c + y_3^2 - 2y_3y_c\]

\[x_1^2 - x_3^2 + y_1^2 - y_3^2 - 2x_c \times (x_1 - x_2) = 2y_c \times (y_1 - y_2)\]

\[y_c = \frac{x_1^2 - x_3^2 + y_1^2 - y_3^2 - 2x_c \times (x_1 - x_3)}{2 \times (y_1 - y_3)}\]

(v) Equate the \(y_c\) from equations (iii) and (iv) and re-arrange so \(x_c\) is alone:
Cancel out the 2’s in the denominators on each side:

\[
\frac{x^2_1 - x^2_2 + y^2_1 - y^2_2 - 2x_c \times (x_1 - x_2)}{2 \times (y_1 - y_2)} = \frac{x^2_1 - x^2_3 + y^2_1 - y^2_3 - 2x_c \times (x_1 - x_3)}{2 \times (y_1 - y_3)}
\]

Cross-multiply each side:

\[
(x^2_1 - x^2_2 + y^2_1 - y^2_2) \times (y_1 - y_3) - 2x_c \times (x_1 - x_2) \times (y_1 - y_3) = (x^2_1 - x^2_3 + y^2_1 - y^2_3) \times (y_1 - y_2) - 2x_c \times (x_1 - x_3) \times (y_1 - y_2)
\]

Re-arrange so \(x_c\) is on one side:

\[
(x^2_1 - x^2_2 + y^2_1 - y^2_2) \times (y_1 - y_3) - (x^2_1 - x^2_3 + y^2_1 - y^2_3) \times (y_1 - y_2) = 2x_c \times [(x_1 - x_2) \times (y_1 - y_3) - (x_1 - x_3) \times (y_1 - y_2)]
\]

\[
\therefore x_c = \frac{(x^2_1 - x^2_2 + y^2_1 - y^2_2) \times (y_1 - y_3) - (x^2_1 - x^2_3 + y^2_1 - y^2_3) \times (y_1 - y_2)}{2 \times [(x_1 - x_2) \times (y_1 - y_3) - (x_1 - x_3) \times (y_1 - y_2)]}
\]

(vi) Repeat steps (iii), (iv) and (v) but getting \(x_c\) alone first and \(y_c\) alone last to get:

\[
y_c = \frac{(y^2_1 - y^2_2 + x^2_1 - x^2_2) \times (x_1 - x_3) - (y^2_1 - y^2_3 + x^2_1 - x^2_3) \times (x_1 - x_2)}{2 \times [(y_1 - y_2) \times (x_1 - x_3) - (y_1 - y_3) \times (x_1 - x_2)]}
\]

(vii) Calculate \(r\) (any of these three equations can be used):

\[
r = \sqrt{(x_1 - x_c)^2 + (y_1 - y_c)^2} = \sqrt{(x_2 - x_c)^2 + (y_2 - y_c)^2} = \sqrt{(x_3 - x_c)^2 + (y_3 - y_c)^2}
\]

Notes:

The equations in steps (v) and (vi) for \(x_c\) and \(y_c\) share common factors:

\[
N_2 = (x^2_1 - x^2_2 + y^2_1 - y^2_2) \quad \quad D_1 = (x_1 - x_2) \times (y_1 - y_3) - (x_1 - x_3) \times (y_1 - y_2)
\]

\[
N_3 = (x^2_1 - x^2_3 + y^2_1 - y^2_3)
\]

Therefore the final equations in steps (v) and (vi) can be simplified:

\[
x_c = \frac{N_2 \times (y_1 - y_3) - N_3 \times (y_1 - y_2)}{2 \times D_1}
\]

\[
y_c = \frac{N_2 \times (x_1 - x_3) - N_3 \times (x_1 - x_2)}{-2 \times D_1} = \frac{N_3 \times (x_1 - x_2) - N_2 \times (x_1 - x_3)}{2 \times D_1}
\]

If the three points are on the same line then \(D_1 = 0\) because there is no circle (it has infinite radius).

If two or more of the three points are the same, then \(N_2 = 0\) and/or \(N_3 = 0\) and there won’t be a solution because three different points are required to calculate a circle’s centre-point and radius.