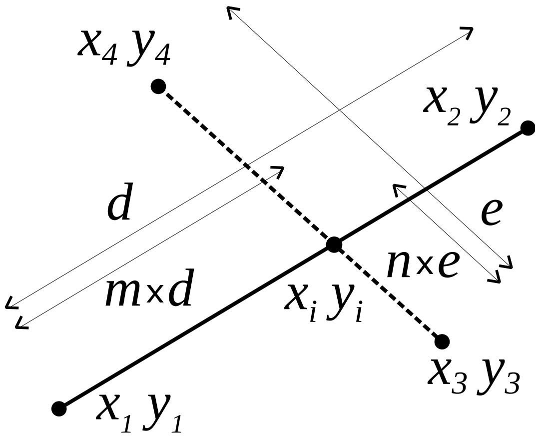
Calculation of Intersection Point of Two Lines and Confirming if Point is within Extents of Lines



Known variables:

x_1y_1 and x_2y_2 = coordinates at each end of *Line 1*
x_3y_3 and x_4y_4 = coordinates at each end of *Line 2*

Unknown variables:

d = length of *Line 1*
e = length of *Line 2*
$x_i\ y_i$ = coordinates at intersection point of two lines
m = ratio along *Line 1* to intersection point
n = ratio along *Line 2* to intersection point

(i) Initial equations:

$$x_i = x_1 + m(x_2 - x_1)$$
$$y_i = y_1 + m(y_2 - y_1)$$

$$x_i = x_3 + n(x_4 - x_3)$$
$$y_i = y_3 + n(y_4 - y_3)$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$e = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$$

(ii) To find m first equate the x_i's and the y_i's and rearrange so n is alone:

$$x_i = x_1 + m(x_2 - x_1) = x_3 + n(x_4 - x_3)$$

$$\therefore n = \frac{(x_1 - x_3) + m(x_2 - x_1)}{(x_4 - x_3)}$$

$$y_i = y_1 + m(y_2 - y_1) = y_3 + n(y_4 - y_3)$$

$$\therefore n = \frac{(y_1 - y_3) + m(y_2 - y_1)}{(y_4 - y_3)}$$

(iii) Next equate the n's and rearrange so m is alone:

$$\frac{(x_1 - x_3) + m(x_2 - x_1)}{(x_4 - x_3)} = \frac{(y_1 - y_3) + m(y_2 - y_1)}{(y_4 - y_3)}$$

$$\therefore (x_1 - x_3)(y_4 - y_3) + m(x_2 - x_1)(y_4 - y_3) = (x_4 - x_3)(y_1 - y_3) + m(x_4 - x_3)(y_2 - y_1)$$

$$\therefore m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{(x_2 - x_1)(y_4 - y_3) - (x_4 - x_3)(y_2 - y_1)}$$

(iv) Similarly, to find n equate the x_i's and the y_i's and rearrange so m is alone:

$$m = \frac{(x_3 - x_1) + n(x_4 - x_3)}{(x_2 - x_1)} = \frac{(y_3 - y_1) + n(y_4 - y_3)}{(y_2 - y_1)}$$

(v) Then equate the m's and rearrange so n is alone:

$$\therefore (x_3 - x_1)(y_2 - y_1) + n(x_4 - x_3)(y_2 - y_1) = (x_2 - x_1)(y_3 - y_1) + n(x_2 - x_1)(y_4 - y_3)$$

$$\therefore n = \frac{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}{(x_4 - x_3)(y_2 - y_1) - (x_2 - x_1)(y_4 - y_3)}$$

Cont'd...

(vi) Rearrange the n equation so that it shares a common denominator with the m equation:

$$D = (x_2 - x_1)(y_4 - y_3) - (x_4 - x_3)(y_2 - y_1)$$

$$\therefore m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{D}$$

and

$$n = \frac{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}{D}$$

This reduces the number of computations required.

(vii) If $D = 0$ then the lines do not intersect (they are parallel).

(viii) The lines intersect within the extents of both lines when:

$$0 \leq m \leq 1 \text{ and } 0 \leq n \leq 1$$

(ix) When using a Ray Casting Algorithm* to determine whether a point is within a polygon made up of contiguous line segments, one of the lines in the above equations will be horizontal:

$$\therefore y_1 = y_2$$

(assuming *Line 1* is the horizontal line and either x_1 or x_2 are the point under examination)

$$\therefore m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{(x_2 - x_1)(y_4 - y_3) - \cancel{(x_4 - x_3)(y_2 - y_1)}}$$

and

$$n = \frac{\cancel{(x_3 - x_1)(y_2 - y_1)} - (x_2 - x_1)(y_3 - y_1)}{(x_2 - x_1)(y_4 - y_3) - \cancel{(x_4 - x_3)(y_2 - y_1)}} = \frac{-\cancel{(x_2 - x_1)}(y_3 - y_1)}{\cancel{(x_2 - x_1)}(y_4 - y_3)}$$

So that:

$$m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{(x_2 - x_1)(y_4 - y_3)} \qquad \text{and} \qquad n = \frac{(y_1 - y_3)}{(y_4 - y_3)}$$

If *Line 2* is also horizontal then $(y_4 - y_3) = 0$ and there is no solution to these equations because the lines do not intersect at a single point.

(* also known as the Crossing Number Algorithm or the Even-Odd Rule Algorithm)