Calculation of Intersection Point of Two Lines and Confirming if Point is within Extents of Lines

Known variables:
- $x_1y_1$ and $x_2y_2$ = coordinates at each end of Line 1
- $x_3y_3$ and $x_4y_4$ = coordinates at each end of Line 2

Unknown variables:
- $d$ = length of Line 1
- $e$ = length of Line 2
- $x_iy_i$ = coordinates at intersection point of two lines
- $m$ = ratio along Line 1 to intersection point
- $n$ = ratio along Line 2 to intersection point

(i) Initial equations:

\[
\begin{align*}
x_i &= x_1 + m(x_2 - x_1) \\
y_i &= y_1 + m(y_2 - y_1)
\end{align*}
\]

\[
\begin{align*}
x_i &= x_3 + n(x_4 - x_3) \\
y_i &= y_3 + n(y_4 - y_3)
\end{align*}
\]

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
e = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}
\]

(ii) To find $m$ first equate the $x_i$’s and the $y_i$’s and rearrange so $n$ is alone:

\[
\begin{align*}
x_i &= x_1 + m(x_2 - x_1) = x_3 + n(x_4 - x_3) \\
y_i &= y_1 + m(y_2 - y_1) = y_3 + n(y_4 - y_3)
\end{align*}
\]

$\therefore n = \frac{(x_1 - x_3) + m(x_2 - x_1)}{(x_4 - x_3)}$ $\therefore n = \frac{(y_1 - y_3) + m(y_2 - y_1)}{(y_4 - y_3)}$

(iii) Next equate the $n$’s and rearrange so $m$ is alone:

\[
\begin{align*}
\frac{(x_1 - x_3) + m(x_2 - x_1)}{(x_4 - x_3)} &= \frac{(y_1 - y_3) + m(y_2 - y_1)}{(y_4 - y_3)}
\end{align*}
\]

$\therefore (x_1 - x_3)(y_4 - y_3) + m(x_2 - x_1)(y_4 - y_3) = (x_4 - x_3)(y_1 - y_3) + m(x_4 - x_3)(y_2 - y_1)$

$\therefore m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{(x_2 - x_1)(y_4 - y_3) - (x_4 - x_3)(y_2 - y_1)}$

(iv) Similarly, to find $n$ equate the $x_i$’s and the $y_i$’s and rearrange so $m$ is alone:

\[
\begin{align*}
m &= \frac{(x_3 - x_1) + n(x_4 - x_3)}{(x_2 - x_1)} = \frac{(y_3 - y_1) + n(y_4 - y_3)}{(y_2 - y_1)}
\end{align*}
\]

(v) Then equate the $m$’s and rearrange so $n$ is alone:

\[
\begin{align*}
\therefore (x_3 - x_1)(y_2 - y_1) + n(x_4 - x_3)(y_2 - y_1) &= (x_2 - x_1)(y_3 - y_1) + n(x_2 - x_1)(y_4 - y_3)
\end{align*}
\]

$\therefore n = \frac{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}{(x_4 - x_3)(y_2 - y_1) - (x_2 - x_1)(y_4 - y_3)}$

Cont’d...
(vi) Rearrange the $n$ equation so that it shares a common denominator with the $m$ equation:

$$D = (x_2 - x_1)(y_4 - y_3) - (x_4 - x_3)(y_2 - y_1)$$

$$\therefore m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{D}$$

and

$$n = \frac{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}{D}$$

This reduces the number of computations required.

(vii) If $D = 0$ then the lines do not intersect (they are parallel).

(viii) The lines intersect within the extents of both lines when:

$$0 \leq m \leq 1 \quad \text{and} \quad 0 \leq n \leq 1$$

(ix) When using a Ray Casting Algorithm* to determine whether a point is within a polygon made up of contiguous line segments, one of the lines in the above equations will be horizontal:

$$\therefore y_1 = y_2$$

(assuming Line 1 is the horizontal line and either $x_1$ or $x_2$ are the point under examination)

$$\therefore m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{(x_2 - x_1)(y_4 - y_3) - (x_4 - x_2)(y_2 - y_1)}$$

and

$$n = \frac{(x_2 - x_1)(y_4 - y_3) - (x_4 - x_1)(y_3 - y_1)}{(x_2 - x_1)(y_4 - y_3) - (x_4 - x_2)(y_2 - y_1)} = \frac{-(x_2 - x_4)(y_3 - y_1)}{(x_2 - x_4)(y_4 - y_3)}$$

So that:

$$m = \frac{(x_4 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_3)}{(x_2 - x_1)(y_4 - y_3)} \quad \text{and} \quad n = \frac{y_1 - y_3}{y_4 - y_3}$$

If Line 2 is also horizontal then $(y_4 - y_3) = 0$ and there is no solution to these equations because the lines do not intersect at a single point.

(* also known as the Crossing Number Algorithm or the Even-Odd Rule Algorithm)