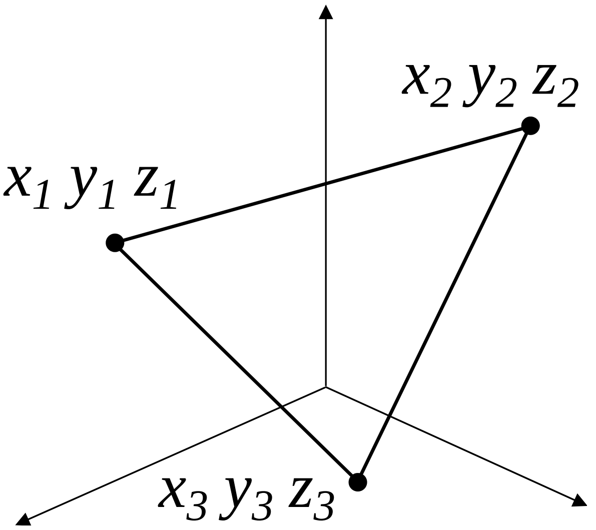
Calculation of 2D Plane and Its Maximum Slope Value/Direction from Three Points on the Plane



Known variables:

Coordinates of three points on plane:

$x_1y_1z_1$, $x_2y_2z_2$ and $x_3y_3z_3$

Unknown variables:

a = intersection of plane on z axis when x and y = 0
b = slope of plane along x axis when y = 0
c = slope of plane along y axis when x = 0

From equation for a 2D plane*:

$z = a + bx + cy$

(* for all planes except those perpendicular to the x, y axes when multiple z values are possible at the same x, y coordinates)

(i) Initial equations:

$$z_1 = a + bx_1 + cy_1$$
$$z_2 = a + bx_2 + cy_2$$
$$z_3 = a + bx_3 + cy_3$$

(ii) Equate the c's to get two equations for a in terms of b:

$$c = \frac{z_1 - a - bx_1}{y_1} = \frac{z_2 - a - bx_2}{y_2} \qquad c = \frac{z_1 - a - bx_1}{y_1} = \frac{z_3 - a - bx_3}{y_3}$$

$$\therefore y_2z_1 - ay_2 - bx_1y_2 = y_1z_2 - ay_1 - bx_2y_1 \qquad \therefore y_3z_1 - ay_3 - bx_1y_3 = y_1z_3 - ay_1 - bx_3y_1$$

$$\therefore a = \frac{y_1z_2 - y_2z_1 + b(x_1y_2 - x_2y_1)}{(y_1 - y_2)} \qquad \therefore a = \frac{y_1z_3 - y_3z_1 + b(x_1y_3 - x_3y_1)}{(y_1 - y_3)}$$

(iii) Equate the a's to get an equation for b:

$$\frac{y_1z_2 - y_2z_1 + b(x_1y_2 - x_2y_1)}{(y_1 - y_2)} = \frac{y_1z_3 - y_3z_1 + b(x_1y_3 - x_3y_1)}{(y_1 - y_3)}$$

$$\therefore (y_1z_2 - y_2z_1)(y_1 - y_3) + b(x_1y_2 - x_2y_1)(y_1 - y_3)$$
$$= (y_1z_3 - y_3z_1)(y_1 - y_2) + b(x_1y_3 - x_3y_1)(y_1 - y_2)$$

$$\therefore (y_1z_2 - y_2z_1)(y_1 - y_3) - (y_1z_3 - y_3z_1)(y_1 - y_2)$$
$$= b[(x_1y_3 - x_3y_1)(y_1 - y_2) - (x_1y_2 - x_2y_1)(y_1 - y_3)]$$

$$\therefore b = \frac{(y_1z_2 - y_2z_1)(y_1 - y_3) - (y_1z_3 - y_3z_1)(y_1 - y_2)}{(x_1y_3 - x_3y_1)(y_1 - y_2) - (x_1y_2 - x_2y_1)(y_1 - y_3)}$$

(iv) Similarly, equate the b's to get two equations for a in terms of c:

$$b = \frac{z_1 - a - cy_1}{x_1} = \frac{z_2 - a - cy_2}{x_2}$$

$$\therefore x_2 z_1 - a x_2 - c x_2 y_1 = x_1 z_2 - a x_1 - c x_1 y_2$$

$$\therefore a = \frac{x_1 z_2 - x_2 z_1 + c(x_2 y_1 - x_1 y_2)}{(x_1 - x_2)}$$

$$b = \frac{z_1 - a - cy_1}{x_1} = \frac{z_3 - a - cy_3}{x_3}$$

$$\therefore x_3 z_1 - a x_3 - c x_3 y_1 = x_1 z_3 - a x_1 - c x_1 y_3$$

$$\therefore a = \frac{x_1 z_3 - x_3 z_1 + c(x_3 y_1 - x_1 y_3)}{(x_1 - x_3)}$$

(v) And equate the a's to get an equation for c:

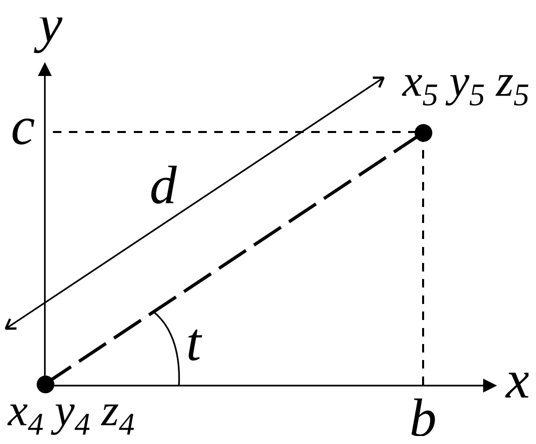
$$\frac{x_1 z_2 - x_2 z_1 + c(x_2 y_1 - x_1 y_2)}{(x_1 - x_2)} = \frac{x_1 z_3 - x_3 z_1 + c(x_3 y_1 - x_1 y_3)}{(x_1 - x_3)}$$

$$\therefore c = \frac{(x_1 z_2 - x_2 z_1)(x_1 - x_3) - (x_1 z_3 - x_3 z_1)(x_1 - x_2)}{(x_3 y_1 - x_1 y_3)(x_1 - x_2) - (x_2 y_1 - x_1 y_2)(x_1 - x_3)}$$

(vi) Finally, use one of the initial equations to calculate a:

$$a = z_1 - b x_1 - c y_1$$

(vii) Use Pythagoras Theorem to determine the direction and value of maximum slope:



Angle to maximum slope in xy plane:

$$t = tan^{-1}(c/b)$$

Distance in xy plane:

$$d = \sqrt{(x_5 - x_4)^2 + (y_5 - y_4)^2}$$

$x_4 = 0$, $y_4 = 0$, $x_5 = b$ and $y_5 = c$

$$\therefore d = \sqrt{b^2 + c^2}$$

Distance in z direction:

$$z_4 = a + b x_4 + c y_4 = a + 0 + 0 = a$$

$$z_5 = a + b x_5 + c y_5 = a + b^2 + c^2$$

$$\therefore z_5 - z_4 = a + b^2 + c^2 - a = b^2 + c^2$$

Value of maximum slope is distance in z direction divided by distance in xy plane:

$$maximum\ slope = \frac{(z_5 - z_4)}{d} = \frac{b^2 + c^2}{(b^2 + c^2)^{1/2}} = \sqrt{b^2 + c^2}$$

Therefore a flat plane has a slope of 0.

Typically this slope value will be multiplied by 100 and expressed as a percentage:

$$\therefore 1\ in\ 40\ slope = 0.025 = 2.5\%$$