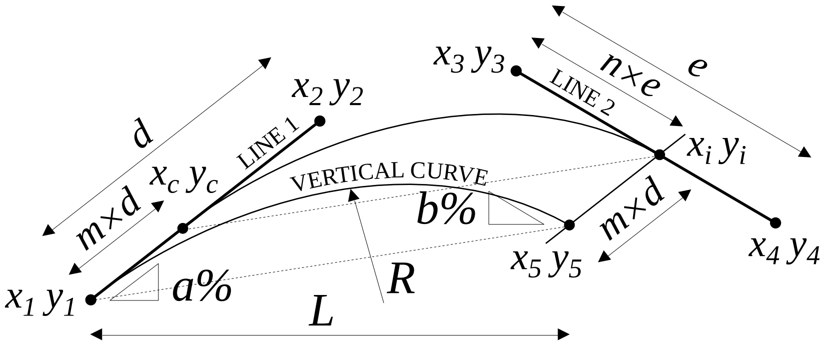
Fitting a Vertical Curve to Two Reference Lines



Known variables:

$x_1\, y_1$ and $x_2\, y_2$ = coordinates at each end of reference line "Line 1"
$x_3\, y_3$ and $x_4\, y_4$ = coordinates at each end of reference line "Line 2"
R = vertical curve radius

Unknown variables:

d = length of Line 1 (don't need to calculate this)
m = fraction along Line 1 to start of vertical curve once it's fitted to two reference lines
$x_C\, y_C$ = coordinates at start of vertical curve once it's fitted to two reference lines
$a\%$ = gradient at start of vertical curve
L = length (along x axis) of vertical curve
$b\%$ = gradient at end of vertical curve
$x_5\, y_5$ = coordinates at end of vertical curve if it starts at $x_1\, y_1$
e = length of Line 2 (don't need to calculate this)
n = fraction along Line 2 to end of vertical curve once it's fitted to two reference lines
$x_i\, y_i$ = coordinates at end of vertical curve once it's fitted to two reference lines

Trying to find:

$x_C\, y_C$ = coordinates at start of vertical curve once it's fitted to two reference lines

Calculations:

(i) Calculate vertical curve parameters:

$$a = 100 \times \frac{(y_2 - y_1)}{(x_2 - x_1)} \qquad\qquad b = 100 \times \frac{(y_4 - y_3)}{(x_4 - x_3)}$$

Rearrange standard vertical curve equation... ...to calculate vertical curve length:

$$R = \frac{100 \times L}{(a - b)} \qquad\qquad L = \frac{R \times (a - b)}{100} \qquad\qquad \therefore x_5 = x_1 + L$$

Use second standard vertical curve equation... ...to calculate level at end of vertical curve:

$$y = y_0 + \frac{a \times x}{100} - \frac{x^2}{2 \times R} \qquad\qquad y_5 = y_1 + \frac{a \times L}{100} - \frac{L^2}{2 \times R}$$

(where y_0 = level at start of vertical curve)

(ii) Calculate fractions along lines:

$$x_i = x_5 + m \times (x_2 - x_1) = x_3 + n \times (x_4 - x_3)$$
$$y_i = y_5 + m \times (y_2 - y_1) = y_3 + n \times (y_4 - y_3)$$

Discard x_i and y_i and rearrange so n items are on one side:

$$n \times (x_4 - x_3) = (x_5 - x_3) + m \times (x_2 - x_1)$$
$$n \times (y_4 - y_3) = (y_5 - y_3) + m \times (y_2 - y_1)$$

Rearrange so n is alone:

$$n = \frac{(x_5 - x_3) + m \times (x_2 - x_1)}{(x_4 - x_3)} = \frac{(y_5 - y_3) + m \times (y_2 - y_1)}{(y_4 - y_3)}$$

Discard n and cross-multiply equations:

$$\therefore (x_5 - x_3) \times (y_4 - y_3) + m \times (x_2 - x_1) \times (y_4 - y_3)$$
$$= (y_5 - y_3) \times (x_4 - x_3) + m \times (y_2 - y_1) \times (x_4 - x_3)$$

Rearrange so m items are on one side:

$$m \times (x_2 - x_1) \times (y_4 - y_3) - m \times (y_2 - y_1) \times (x_4 - x_3)$$
$$= (y_5 - y_3) \times (x_4 - x_3) - (x_5 - x_3) \times (y_4 - y_3)$$

Rearrange so m is alone:

$$m = \frac{(y_5 - y_3) \times (x_4 - x_3) - (x_5 - x_3) \times (y_4 - y_3)}{(x_2 - x_1) \times (y_4 - y_3) - (y_2 - y_1) \times (x_4 - x_3)}$$

(iii) Calculate coordinates at start of vertical curve once it's fitted to two reference lines:

$$x_C = x_1 + m \times (x_2 - x_1)$$
$$y_C = y_1 + m \times (y_2 - y_1)$$

Vertical curve can now be fitted to two reference lines.